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LETTER TO THE EDITOR

Geometric amplitude factor in an *LCR* circuit with time-dependent inductance, capacitance and resistance

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Abstract
The geometric amplitude factor in an over-damped *LCR* circuit with time-dependent inductance, capacitance and resistance is obtained. A simple scheme to measure this factor is also proposed.

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The discovery of geometric phases [1] is one of the most important advances in physics during the past 20 years. After this profound discovery, Berry investigated geometric amplitude factors in adiabatic quantum transitions [2]. Geometric phases have been found in many physical systems theoretically and experimentally [3]. Recently, it was shown, based on the similarity between mechanical and electromagnetic oscillations, that there exists a Hannay angle (classical geometric phase [4, 5]) in an under-damped *LCR* circuit with time-dependent parameters [6]. Here, we show that there exists a geometric amplitude factor in an over-damped *LCR* circuit when the inductance, capacitance and resistance of the circuit change with time periodically and adiabatically.

First, let us briefly review the well-known results for an *LCR* circuit with time-independent inductance, capacitance and resistance, denoted by L_0, C_0 and R_0, respectively. The differential equation for such a circuit is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L_0\frac{\mathrm{d}Q}{\mathrm{d}t}\right) + R_0\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C_0} = 0. \tag{1}$$

In the case of under-damping (with $4L_0C_0 > (C_0R_0)^2$), the solution is given by

$$Q(t) = \exp\left(-\frac{R_0}{2L_0}t\right)\{A\exp[\mathrm{i}\varphi_D(t)] + B\exp[-\mathrm{i}\varphi_D(t)]\} \tag{2}$$

where A and B are some constants, $\varphi_D(t) = \omega_D t$ and $\omega_D = \sqrt{1/L_0C_0 - (R_0/2L_0)^2}$. In the case of over-damping (with $4L_0C_0 < (C_0R_0)^2$), the solution can be written as

$$Q(t) = \exp\left(-\frac{R_0}{2L_0}t\right)\{A\exp[-\mathrm{Im}\,\varphi_D(t)] + B\exp[+\mathrm{Im}\,\varphi_D(t)]\} \tag{3}$$

where $\mathrm{Im}\,\varphi_D(t)$ denotes the imaginary part of $\varphi_D(t)$. It is obvious that solution (3) could be directly obtained from solution (2) if we regard the ω_D in the latter as an imaginary number.

Now we turn to the *LCR* circuit with time-dependent inductance, capacitance and resistance. The differential equation for such a circuit is [6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\frac{\mathrm{d}Q}{\mathrm{d}t}\right)+R(t)\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{Q}{C(t)}=0. \tag{4}$$

In the following, we will always assume that the parameters of the circuit change with time adiabatically and periodically. In the case of under-damping, the solution of equation (4) can approximately be expressed as [5]

$$Q(t)\approx\rho(t)\exp\left(-\int_0^t\frac{R}{2L}\,\mathrm{d}t\right)\{A\exp[\mathrm{i}\varphi_D(t)+\mathrm{i}\Delta\theta(t)]+B\exp[-\mathrm{i}\varphi_D(t)-\mathrm{i}\Delta\theta(t)]\} \tag{5}$$

where $\rho(t)\approx[\omega_D(t)]^{-1/2}$, $\varphi_D(t)=\int_0^t\omega_D(t)\,\mathrm{d}t$ and $\Delta\theta(t)=\int_0^t\left(-\frac{1}{4L\omega_D}\cdot\frac{\mathrm{d}R}{\mathrm{d}t}\right)\mathrm{d}t$. The charge on the capacitor plate after one period is

$$\begin{aligned}Q(T)\approx\rho(T)\exp\left(-\int_0^T\frac{R}{2L}\,\mathrm{d}t\right)&\{A\exp[\mathrm{i}\varphi_D(T)+\mathrm{i}\Delta\theta(T)]\\&+B\exp[-\mathrm{i}\varphi_D(T)-\mathrm{i}\Delta\theta(T)]\}.\end{aligned} \tag{6}$$

Solution (6) suggests that there exists a geometric phase (Hannay angle)

$$\Delta\theta(T)=\oint-\frac{1}{4L\omega_D}\,\mathrm{d}R=\oint-\frac{1}{4\sqrt{L/C-(R/2)^2}}\,\mathrm{d}R \tag{7}$$

besides a dynamical phase $\varphi_D(T)=\int_0^T\omega_D\,\mathrm{d}t$, in the total phase of Q [6]. In the following discussion we will show the existence of a geometric amplitude factor in the case of over-damping, which is the main purpose of this letter.

If the condition $4LC<(CR)^2$ is satisfied, equation (4) describes an over-damped time-dependent *LCR* circuit, and the solution can approximately be expressed as

$$\begin{aligned}Q(t)\approx\rho'(t)\exp\left(-\int_0^t\frac{R}{2L}\,\mathrm{d}t\right)&\{A\exp[-\mathrm{Im}\,\varphi_D(t)-\mathrm{Im}\,\Delta\theta(t)]\\&+B\exp[\mathrm{Im}\,\varphi_D(t)+\mathrm{Im}\,\Delta\theta(t)]\}\end{aligned} \tag{8}$$

where $\rho'(t)=[\mathrm{Im}\,\omega_D(t)]^{-1/2}$. The charge on the capacitor plate after one period is

$$\begin{aligned}Q(T)\approx\rho'(T)\exp\left(-\int_0^T\frac{R}{2L}\,\mathrm{d}t\right)&\{A\exp[-\mathrm{Im}\,\varphi_D(T)-\mathrm{Im}\,\Delta\theta(T)]\\&+B\exp[\mathrm{Im}\,\varphi_D(T)+\mathrm{Im}\,\Delta\theta(T)]\}.\end{aligned} \tag{9}$$

Solution (9) can be rewritten in a more suggestive form:

$$Q(T)\approx\rho'(T)\exp\left(-\int_0^T\frac{R}{2L}\,\mathrm{d}t\right)\left(A\Gamma_D^{-1}\Gamma_G^{-1}+B\Gamma_D\Gamma_G\right) \tag{10}$$

where

$$\Gamma_D=\exp[\mathrm{Im}\,\varphi_D(T)]=\exp\left[\int_0^T\sqrt{(R/2L)^2-1/LC}\,\mathrm{d}t\right] \tag{11}$$

$$\Gamma_G=\exp[\mathrm{Im}\Delta\theta(T)]=\exp\left[-\oint\frac{1}{4\sqrt{(R/2)^2-L/C}}\,\mathrm{d}R\right]. \tag{12}$$

If we regard Γ_D as a dynamical amplitude factor corresponding to the dynamical phase factor in the case of under-damping, Γ_G can be recognized as a geometric amplitude factor, which is only dependent on the path in the parameter space, corresponding to the geometric phase factor in the case of under-damping. From the foregoing analysis, we can conclude that there exists a geometric amplitude factor, besides a dynamical amplitude factor, in an over-damped *LCR* circuit when the parameters of the circuit change with time periodically and adiabatically.

This factor can be measured in the experiment. While the scheme proposed in [6] for the measurement of Hannay angle in the under-damped *LCR* circuit is also valid for measuring the geometric amplitude factor, here we propose a different one.

Consider a time-dependent over-damped *LCR* circuit described by equation (4) with initial conditions $Q(0) = Q_0$ and $I(0) = 0$. It is easy to prove that with such initial conditions constants A and B in solution (8) satisfy the inequality $|A| < |B|$, and in the adiabatical limit solution (10) can be simplified as

$$Q(T) \approx B\rho'(T) \exp\left(-\int_0^T \frac{R}{2L}\,\mathrm{d}t\right)\Gamma_D\Gamma_G. \tag{13}$$

The adiabatically and periodically varying parameters of the circuit determine a closed path in the parameter space. It is heuristic to study how Γ_D and Γ_G change when the path is reversed (that is $L(t)$,$C(t)$ and $R(t)$ are replaced by $L(-t)$,$C(-t)$ and $R(-t)$, respectively) [2]. Let P_+ and P_- denote a given closed path and its reversal in the parameter space, respectively. One can observe that

$$\Gamma_D(P_+) = \Gamma_D(P_-) \qquad \Gamma_G(P_+) = [\Gamma_G(P_-)]^{-1}. \tag{14}$$

This fact leads to a simple scheme to measure the geometric amplitude factor.

Consider two time-dependent *LCR* circuits with the same initial conditions, i.e. $Q_1(0) = Q_2(0) = Q_0$ and $I_1(0) = I_2(0) = 0$. Assume that the adiabatically and periodically varying parameters of the first circuit determine a closed path P_+ and those of the second determine a closed path P_- (the reverse of P_+) in the parameter space. From the foregoing analysis, one can find that if $Q_1(T)$ and $Q_2(T)$ are measured in the experiment, the geometric amplitude factors acquired in one period in the first circuit and in the second are given by

$$\Gamma_{G1} = \Gamma_G(P_+) = [Q_1(T)/Q_2(T)]^{1/2} \qquad \Gamma_{G2} = \Gamma_G(P_-) = [Q_2(T)/Q_1(T)]^{1/2}. \tag{15}$$

It should be noted that this scheme had been proposed for the measurement of geometric amplitude factor in the adiabatical quantum transition in [2], and it could also be used to measure the Hannay angle in the case of under-damping.

In summary, the geometric amplitude factor in an over-damped *LCR* circuit with time-dependent inductance, capacitance and resistance has been obtained in this letter. A simple scheme to measure this factor has also been proposed. We expect experimental observations of the geometric amplitude factor in the time-dependent *LCR* circuit.

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